Stability of fixed points in the $(4 + \epsilon)$-dimensional random field O(N) spin model for sufficiently large N

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We study the stability of fixed points in the two-loop renormalization group for the random field O(N) spin model in $4 + \epsilon$ dimensions. We solve the fixed-point equation in the $1/N$ expansion and ϵ expansion. In the large-N limit, we study the stability of all fixed points. We solve the eigenvalue equation for the infinitesimal deviation from the fixed points under physical conditions on the random anisotropy function. We find that the fixed point corresponding to dimensional reduction is singly unstable and others are unstable or unphysical. Therefore, one has no choice other than dimensional reduction in the large-N limit. The two-loop β function enables us to find a compact area in the (d, N) plane where the dimensional reduction breaks down. We calculate higher-order corrections in the $1/N$ and ϵ expansions to the fixed point. Solving the corrected eigenvalue equation nonperturbatively, we find that this fixed point is singly unstable also for sufficiently large N and the critical exponents show a dimensional reduction.

I. INTRODUCTION

The random field O(N) spin model is one of the simplest models with both quenched disorder and spin correlations. It is a fundamental problem to clarify the critical phenomena in this model. Dimensional reduction[1] is one key to solve this problem. Dimensional reduction claims that the critical behavior of the random field O(N) spin model in d space dimensions is the same as that of the pure O(N) spin model in $d - 2$ space dimensions. If this conjecture is true, all critical exponents of the random field spin model in d dimensions should be identical to those of the corresponding pure model in $d - 2$ dimensions.

Since several rigorous results for the random field Ising model ($N = 1$ case) indicated the failure of dimensional reduction to predict the lower critical dimensions,[2,3,4] the breakdown of dimensional reduction with some approximation methods was discussed in order to obtain intuitive understanding or quantitative information. Fisher calculated the one-loop renormalization group, and he pointed out the breakdown of dimensional reduction due to the appearance of an infinite number of relevant operators in $4 + \epsilon$ dimensions.[5] He showed the existence of a fixed point corresponding to the dimensional reduction for $N \geq 18$, but he argued that this fixed point should be unstable as far as the number of spin components, N, is finite. Therefore, he concluded that the dimensional reduction was not valid near four dimensions. Feldman

found a nonanalytic fixed point in Fisher's renormalization group for several small N.[6] He obtained nontrivial critical exponents shifted from the predictions of dimensional reduction. Mézard and Young also suggested the breakdown of dimensional reduction by replica symmetry breaking.[7] Now, many researchers believe that dimensional reduction is incorrect in dimensions lower than 6.

Recently, Tarjus and Tissier studied the critical phenomena of this model in any dimensions and for any value of N by using the nonperturbative renormalization group method and the replica method.[8] They show the following relation of the critical exponents of the two-point spin correlation function:

$$\eta = \bar{\eta} = \frac{\epsilon}{N - 2}, \tag{1}$$

predicted by dimensional reduction in a certain region in the (d, N) plane. Since this relation seems valid for $N \geq 18$ near four dimensions, the consistency between their result and that of the Refs. 5 and 7 should be studied.

To understand the consistency of their works, the de Almeida–Thouless criterion[9] is applied faithfully to this model in a simple $1/N$-expansion method.[10] It is shown that the saddle point of the auxiliary field is stable in the random field O(N) spin model. Also the stability argument by Balents and Fisher for random media[11] is applied to Fisher's one-loop renormalization group; then, the following two possibilities are indicated.[10] The fixed point corresponding to dimensional reduction is singly

unstable, or there is no singly unstable fixed point. Combining these results leads to relation (1) which is the same result obtained by Tarjus and Tissier.

In this paper, we study the stability of the fixed point corresponding to dimensional reduction. Particularly, we discuss the physical condition on the deviation from the fixed point. Since Fisher did not solve the eigenvalue problem completely for the stability around the fixed point,[5] we solve this problem in a $1/N$ expansion. In the large-N limit, the stability of all fixed points is studied. The solution of eigenvalue equations for the deviation from the fixed points indicates that the only once unstable fixed point shows the critical exponents predicted by dimensional reduction. Next, we study this fixed point by the two-loop renormalization group obtained by Le Doussal and Wiese[12] and Tissier and Tarjus.[13] The double expansion in $1/N$ and ϵ enables us to calculate the correction to the fixed point. The solution of the eigenvalue equation shows that the unstable modes pointed out by Fisher are fictitious. Therefore, we conclude that the fixed point yielding relation (1) is singly unstable for sufficiently large N. This result agrees with that obtained by Tarjus and Tissier[8] and also by the simple $1/N$ expansion.[10] Furthermore, we calculate higher-order corrections to the eigenvalues.

This paper is organized as follows. In Sec. II, we briefly review Fisher's renormalization group analysis for the random field O(N) spin model in $4 + \epsilon$ dimensions.[5] In Sec. III, we discuss possible singularities of the random anisotropy functions for a physical model. In Sec. IV, we treat the one-loop renormalization group in the large-N limit. In this limit, the critical phenomena in $4 + \epsilon$ dimensions are shown to be governed by the fixed point which gives the result of dimensional reduction. In Sec. V, we investigate the stability of this fixed point corresponding to dimensional reduction by the two-loop renormalization group. We show that this fixed point is singly unstable on the basis of the physical condition on the coupling function discussed in Sec. III. Thus, we conclude that the prediction of dimensional reduction for the critical exponents (1) holds for sufficiently large N. In Sec. VI, we calculate higher-order corrections to the eigenvalue and the exponents of the singularities. In Sec. VII, we summarize our results and discuss some problems.

II. FISHER'S ONE-LOOP RENORMALIZATION GROUP ANALYSIS

In this section, we briefly review Fisher's argument on the instability of the fixed point corresponding to the dimensional reduction in the one-loop renormalization group for the random field O(N) spin model in $4 + \epsilon$ dimensions.[5]

A. Model

We consider O(N) classical spins $\boldsymbol{S}(x)$ with a fixed-length constraint $\boldsymbol{S}(x)^2 = 1$. To take the average over the random field, one introduces replicas $\boldsymbol{S}^\alpha(x)$, $\alpha = 1, \ldots, n$. We start from a nonlinear σ model of the following replica partition function and action:

$$\mathcal{Z} = \int \prod_{\alpha=1}^n \mathcal{D}\boldsymbol{S}^\alpha \delta(\boldsymbol{S}^{\alpha 2} - 1)e^{-\beta H_{\rm rep}},$$

$$\beta H_{\rm rep} = \frac{a^{2-d}}{2T} \int d^d x \sum_{\alpha=1}^n \sum_{\mu=1}^d (\partial_\mu \boldsymbol{S}^\alpha)^2$$
$$-\frac{a^{-d}}{2T^2} \int d^d x \sum_{\alpha,\beta}^n R(\boldsymbol{S}^\alpha \cdot \boldsymbol{S}^\beta), \qquad (2)$$

where a is the ultraviolet cutoff and the parameter T denotes the dimensionless temperature. The function $R(\boldsymbol{S}^\alpha \cdot \boldsymbol{S}^\beta)$ represents general anisotropy including the random field and all the random anisotropies, and is given by

$$R(\boldsymbol{S}^\alpha \cdot \boldsymbol{S}^\beta) = \sum_{\mu=1}^\infty \Delta_\mu (\boldsymbol{S}^\alpha \cdot \boldsymbol{S}^\beta)^\mu, \qquad (3)$$

where Δ_μ denotes the strength of the random field and the μth rank random anisotropy ($\mu = 1$ is the random field, and $\mu \geq 2$ is the second- and higher-rank random anisotropy). These coupling constants are positive semidefinite $\Delta_\mu \geq 0$.

B. One-loop β function

The β function $\partial_t R(z)$ at zero temperature can be expressed in the loop expansion

$$\partial_t R(z) = \beta_0[R] + \beta_1[R] + \beta_2[R] + \cdots. \qquad (4)$$

Here, we have defined the scale parameter t which increases toward the infrared direction. Fisher calculated the one-loop β function in the following form

$$\beta_0[R] = -\epsilon R(z), \qquad (5)$$
$$\beta_1[R] = 2(N-2)R'(1)R(z) - (N-1)zR'(1)R'(z)$$
$$+ (1 - z^2)R'(1)R''(z) + \frac{1}{2}R'(z)^2(N - 2 + z^2)$$
$$- R'(z)R''(z)z(1 - z^2) + \frac{1}{2}R''(z)^2(1 - z^2)^2. \qquad (6)$$

Expanding $R(z)$ around $z = 1$, we obtain the one-loop β functions for $R'(1)$ and $R''(1)$:

$$\partial_t R'(1) = -\epsilon R'(1) + (N-2)R'(1)^2, \qquad (7)$$
$$\partial_t R''(1) = -\epsilon R''(1) + 6R'(1)R''(1)$$
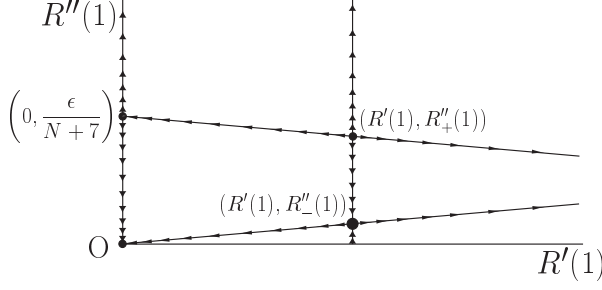$$+ (N+7)R''(1)^2 + R'(1)^2. \qquad (8)$$

FIG. 1: The renormalization group flow for the couplings $R'(1)$ and $R''(1)$. The fixed point $(0, \frac{\epsilon}{N+7})$ is unphysical, because the corresponding Δ_1 becomes negative. See Sec. IV A.

The β functions (8) and (8) have two nontrivial fixed points (see Fig. 1):

$$(R'(1), R''_+(1))$$
$$= \left(\frac{\epsilon}{N-2}, \frac{(N-8) + \sqrt{(N-2)(N-18)}}{2(N-2)(N+7)} \epsilon \right), \quad (9)$$

$$(R'(1), R''_-(1))$$
$$= \left(\frac{\epsilon}{N-2}, \frac{(N-8) - \sqrt{(N-2)(N-18)}}{2(N-2)(N+7)} \epsilon \right). (10)$$

The formulas for the critical exponents η and $\bar{\eta}$,

$$\eta = R'(1), \quad \bar{\eta} = (N-1)R'(1) - \epsilon, \qquad (11)$$

enable us to obtain the correlation function critical exponents (1). This result confirms one of the predictions by dimensional reduction. From the fixed points (9) and (10), we find that these results are applicable only for $N \geq 18$. The eigenvalues $\epsilon\lambda_1$ and $\epsilon\lambda_2^{\pm}$ of the scaling matrix at the fixed points (9) and (10) are given by

$$\lambda_1 = +1, \qquad (12)$$
$$\lambda_2^{\pm} = \pm\sqrt{\frac{N-18}{N-2}}. \qquad (13)$$

Thus, the fixed point (9) is unstable. The fixed point (10) seems to be stable for $N \geq 18$.

However, Fisher claimed an instability of this fixed point (10). For the one-loop β function in terms of differential coefficients $R^{(k)}(1)$, ($k = 0, 1, 2, \ldots$), he obtained a triangular scaling matrix at the fixed point $(R'(1)^*, R''_-(1)^*, \ldots, R^{(k-1)}(1)^*, \ldots)$, whose diagonal components were given in the following series:

$$\lambda_k = \frac{2k^2 - k(N-1) + 2N - 4}{N-2} - 1 + kNR''_-(1)^*$$
$$\simeq 1 - k + \frac{2k^2}{N} \quad (k \geq 3). \qquad (14)$$

Almost all eigenvalues were indicated to be positive for a sufficiently large k, although one should add a term

$2nkP_2P_k$ missed in Eq. (C6) of his paper.[5] Then, Fisher concluded that there was no singly unstable fixed point and dimensional reduction broke down near four dimensions. In Sec. V, however, we show that these infinitely many relevant modes are unphysical by solving this eigenvalue problem completely. In the following sections, we carefully reexamine the renormalization group.

III. ALLOWED SINGULARITIES OF THE RANDOM ANISOTROPY FUNCTION

The fixed-point condition of the renormalization group determines properties of the function $R(z)$. Here we discuss possible asymptotic behaviors of $R(z)$ near $z = \pm 1$. The first derivative of the fixed-point equation with respect to z is

$$[-\epsilon + (N-3)R'(1)]R'(z) + zR'(z)^2$$
$$-(N+1)zR'(1)R''(z) + (N-3+4z^2)R'(z)R''(z)$$
$$+(1-z^2)R'(1)R'''(z) - z(1-z^2)R'(z)R'''(z)$$
$$-3z(1-z^2)R''(z)^2 + (1-z^2)^2R''(z)R'''(z) = 0.(15)$$

If we assume the asymptotic behavior of $R'(z)$ near $z = 1$,

$$R'(z) = R'(1) + C(1-z)^{\alpha} + \cdots, \qquad (16)$$

with $0 < \alpha$. To discuss a cuspy behavior of $R(z)$ at $z = 1$, we consider only $\alpha < 1$. The condition (15) gives the following constraint:

$$[-\epsilon + (N-2)R'(1)]R'(1)$$
$$-C^2\alpha(4\alpha^2 + 4\alpha + N - 1)(1-z)^{2\alpha-1} = 0. \quad (17)$$

For $\alpha \neq 1/2$, this constraint gives

$$\alpha = \frac{1}{2}(-1 + \sqrt{2-N}) \qquad (18)$$

or

$$C = 0, \qquad (19)$$

and also

$$R'(1) = \frac{\epsilon}{N-2} \qquad (20)$$

or

$$R'(1) = 0. \qquad (21)$$

Here, the former case given by Eqs.(19) and (20) shows the dimensional reduction. The formulas for (11), the critical exponents obtained by Feldman,[6] enable us to obtain the critical exponents

$$\eta = \frac{\epsilon}{N-2} = \bar{\eta}.$$

Therefore, no $\alpha \neq 1/2$ is allowed for any $N > 2$. For $\alpha = 1/2$, the parameter $R'(1)$ can change continuously

depending on the constant C. Therefore, only $\alpha = 1/2$ allows divergent $R''(1)$. Only in this case does the non-trivial critical behavior differ from the prediction of dimensional reduction. Since the initial value $R(z)$ of the renormalization group equation (6) is an analytic function, the flow of $R''(1)$ should diverge for the breakdown of dimensional reduction.

The same discussion for $z = -1$ can be done. The only possible singularity is

$$R'(z) = R'(-1) + C(1 + z)^{1/2} + \cdots,$$

with $C = -R'(1)$.

Next, we consider the renormalization group flow of the singular function $R'(z)$. If we assume an initial coupling

$$R'(z) = C(1 - z)^{\alpha}, \qquad (22)$$

with $\alpha > 0$, the renormalization group transformation generates a term $-C^2(1 - z)^{2\alpha - 1}$. If $2\alpha - 1 < \alpha$–namely, $\alpha < 1$–the successive transformations produce less power. Eventually, the flow generates a term

$$R'(z) \sim (1 - z)^{\alpha'},$$

with $\alpha' < 0$, unless $\alpha = \frac{1}{2}$ or $\alpha \geq 1$. To avoid the flow to such unphysical regions, we require a condition

$$\alpha = \frac{1}{2} \quad \text{or} \quad \alpha \geq 1 \qquad (23)$$

on the initial function (22). Therefore, the allowed function has the same singularity as that of the fixed-point function.

IV.   LARGE-N LIMIT

Here, we take the large-N limit in the one-loop renormalization group with $NR(z)$ finite and redefine $R(z)$ by $NR(z) \to R(z)$. The one-loop β function for $R(z)$ becomes

$$\partial_t R(z) = -\epsilon R(z) + 2R'(1)R(z) - zR'(1)R'(z)$$
$$+ \frac{1}{2}R'(z)^2. \qquad (24)$$

A.   Fixed points

Following the method given by Balents and Fisher,[11] we consider the flow equation for $R'(z)$ instead of that for $R(z)$. Differentiating the one-loop β function with respect to z, we have

$$\partial_t R'(z) = -\epsilon R'(z) + R'(1)R'(z) - zR'(1)R''(z)$$
$$+ R'(z)R''(z). \qquad (25)$$

We redefine the parameters

$$R'(z) \equiv \epsilon u(z), \quad t' \equiv \epsilon t, \quad u(1) \equiv a, \qquad (26)$$

and we consider the fixed-point equation

$$0 = (a - 1)u(z) - zau'(z) + u(z)u'(z). \qquad (27)$$

Substituting $z = 1$ into Eq. (27), we have two cases

$$a = 0, 1, \qquad (28)$$

for finite $u'(1)$. Solving the differential equation (27) for $a = 1$, we have two nontrivial solutions:

$$R'(z) = \epsilon \qquad (29)$$

and

$$R'(z) = \epsilon z. \qquad (30)$$

The first one indicates

$$(\Delta_1, \Delta_2) = (\epsilon, 0), \qquad (31)$$
$$(R'(1), R''(1)) = (\epsilon, 0). \qquad (32)$$

Thus, the solution (29) is the "random field solution." The second one indicates

$$(\Delta_1, \Delta_2) = \left(0, \frac{\epsilon}{2}\right), \qquad (33)$$
$$(R'(1), R''(1)) = (\epsilon, \epsilon). \qquad (34)$$

Thus, the solution (30) is not the "random field solution" but the "random anisotropy solution."

For $a = 0$, the nontrivial solution is

$$R'(z) = \epsilon(z - 1). \qquad (35)$$

This indicates

$$(\Delta_1, \Delta_2) = \left(-\epsilon, \frac{\epsilon}{2}\right), \qquad (36)$$
$$(R'(1), R''(1)) = (0, \epsilon). \qquad (37)$$

Thus, the solution (35) is unphysical. This fact indicates that the trivial fixed point $R'(z) = 0$ is the unique physical solution when $a = 0$.

If $a \neq 0, 1$,

$$\frac{du(z)}{dz} = \frac{(a - 1)u(z)}{za - u(z)}. \qquad (38)$$

Taking the inversion, we regard z as a function of u. One gets

$$\frac{dz(u)}{du} = \frac{a}{a - 1}\frac{z(u)}{u} - \frac{1}{a - 1}, \qquad (39)$$

which is easily integrated. Then, we have

$$z = u - (a - 1)\left|\frac{u}{a}\right|^{a/(a-1)}. \qquad (40)$$

Because $z(u)$ takes the maximum value 1 at $u = a$, $u(z)$ is double valued as we show in Fig. 2. It is seen from Eq. (38) that du/dz is ill defined on $u = az$. Therefore the
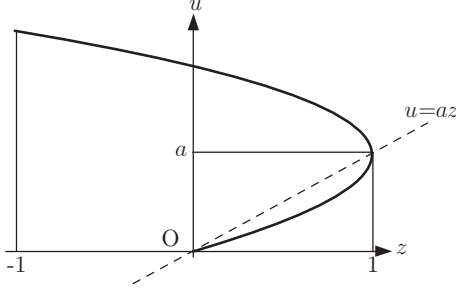
FIG. 2: A schematic graph of $u(z)$. Since the derivative of u is ill defined on $u = az$, the solution terminates on this line. The above graph represents two solutions meeting at $(1, a)$.

lower branch terminates at the origin, so that it should be continued to the region $-1 \leq z < 0$. This is possible only if $a/(a-1) = n$ is a positive integer. In this case, we have $a = \frac{n}{n-1} > 1$.

Expanding u around a, we have

$$
\begin{aligned}
z &= u - (a-1)\left|\frac{u}{a}\right|^{a/(a-1)} \\
&= a + (u-a) - (a-1)\left(1 + \frac{u-a}{a}\right)^{a/(a-1)} \\
&\simeq 1 - \frac{1}{2a(a-1)}(u-a)^2.
\end{aligned}
\tag{41}
$$

Since $-1 \leq z \leq 1$, we have

$$
1 - z \simeq \frac{(u-a)^2}{2a(a-1)} \geq 0.
\tag{42}
$$

Thus, we find that the fixed point a must satisfy $a(a-1) > 0$. Since $a < 0$ gives a negative exponent $\epsilon + \bar{\eta} < 0$ of the disconnected correlation function, $a < 0$ is excluded and we have $a > 1$. The Schwartz-Soffer inequality[14] $2\eta \geq \bar{\eta}$ also requires the other constraint $a \leq 1$. Therefore, these nonanalytic solutions (40) are unphysical.

B. Stabilities of the fixed-point solutions

Next, we study the stability of the fixed points. Let $u(z)$ be a fixed-point solution:

$$
0 = u(z)(a-1) + u'(z)(u(z) - az).
\tag{43}
$$

We consider an infinitesimal deformation $u(z) \rightarrow u(z) + v(z)$ and $a \rightarrow a + b$, with a finiteness condition $\sup_{-1 \leq z \leq 1} |v(z)| < \infty$ and $b < \infty$. We study the behavior of the first order in $v(z)$ and b:

$$
\begin{aligned}
&v(z)(a-1) + u(z)b + v'(z)[u(z) - az] \\
&+ u'(z)[v(z) - bz] = \lambda v(z).
\end{aligned}
\tag{44}
$$

$\epsilon\lambda$ denotes the eigenvalue of the scaling operator. The negative eigenvalue $\epsilon\lambda < 0$ indicates that the fixed-point

solution is stable, and the positive eigenvalue $\epsilon\lambda > 0$ indicates that the fixed-point solution is unstable. Normalizing $v(z)$ appropriately, we can take $v(1) = 0$ or $v(1) = 1$.

1. $R'(z) = \epsilon$

For $a = 1$ and $u(z) = 1$, Eq. (44) becomes

$$
b + v'(z)(1-z) = \lambda v(z),
\tag{45}
$$

where b represents $v(1)$ taking 0 or 1. When $b = 0$, the solution is

$$
v(z) = C(1-z)^{-\lambda},
\tag{46}
$$

where $\lambda < 0$ because of the initial condition $b = v(1) = 0$. On the other hand, when $b = 1$, a general solution is

$$
v(z) = \begin{cases} \lambda^{-1} + c(1-z)^{-\lambda} & (\lambda \neq 0), \\ \ln|1-z| & (\lambda = 0). \end{cases}
\tag{47}
$$

Here the condition $b = 1$ requires that $\lambda = 1$ and $c = 0$. Since the singularity of the physical deformation $v(z)$ satisfies the condition (22), the allowed value of λ is

$$
\lambda = -\frac{1}{2}, \quad \lambda \leq -1 \quad \text{or} \quad \lambda = 1.
\tag{48}
$$

This shows that the fixed-point solution is singly unstable. Note that the cuspy deformation from this fixed point is irrelevant.

2. $R'(z) = \epsilon z$

For $a = 1$ and $u(z) = z$, Eq. (44) becomes

$$
v(z) = \lambda v(z).
\tag{49}
$$

Then, $\lambda = 1$, and the fixed point is fully unstable. Note that this instability occurs for large N.

3. $R(z) = 0$

Since $a = 0$ and $u = 0$ in this case, Eq. (44) is $-v(z) = \lambda v(z)$, which means $\lambda = -1$ for any $v(z)$; thus the trivial fixed point is fully stable.

V. TWO-LOOP RENORMALIZATION GROUP ANALYSIS

Here, we discuss the stability of the fixed point by solving the eigenvalue problem of the scaling operator in the two-loop renormalization group.

A. The two-loop β function

Recently, Le Doussal and Wiese calculated a two-loop β function at zero temperature. Independently, Tarjus and Tissier obtained the following consistent result

$$
\begin{aligned}
\beta_2[R] \\
= \frac{1}{2}(N-2)\{&(1-z^2)^2 R''(z)^3 \\
&-(1-z^2)[3zR'(z) - (2+z^2)R'(1)]R''(z)^2 \\
&-2(1-z^2)[R'(z) - zR'(1)]R'(z)R''(z) \\
&+(1-z^2)R'(1)R'(z)^2 + 4R'(1)^2 R(z)\} \\
-\frac{1}{2}&(1-z^2)[(1-z^2)R'''(z) - 3zR''(z) - R'(z)]^2 \\
\times&[-(1-z^2)R''(z) + zR'(z) - R'(1)] \\
-\frac{c^2}{2}&[(N+2)(1-z^2)R''(z) - (3N-2)zR'(z) \\
&+8K(N-2)R(z)],
\end{aligned} \tag{50}
$$

where $c = \lim_{z \nearrow 1} \sqrt{1-z^2} R''(z)$ and $K = 2\gamma_a$ is an unknown real number. As discussed in the one-loop β function in Sec. III, the possible singularity in the fixed-point function $R'(z) \sim (1-z)^\alpha$ and the possible deformation from the fixed point are given by $\alpha = \frac{1}{2}$ or $\alpha \geq 1$. Otherwise, the function cannot be a fixed point. This condition is preserved exactly also in the two-loop β function. In the following, we present some corrected results based on the two-loop renormalization group.

B. Fixed point in the two-loop β function

First, we discuss a condition on N and d for the existence of the fixed point corresponding to the dimensional reduction. The two-loop β functions for the two coupling constants are

$$
\begin{aligned}
\partial_t R'(1) =\ & -\epsilon R'(1) + (N-2)R'(1)^2 \\
& + (N-2)R'(1)^3, \\
\partial_t R''(1) =\ & -\epsilon R''(1) + (N+7)R''(1)^2 + 6R''(1)R'(1) \\
& + R'(1)^2 + 2(5N+17)R''(1)^3 \\
& + 6(N+7)R''(1)^2 R'(1) \\
& - 6(N-5)R''(1)R'(1)^2 - (N-4)R'(1)^3.
\end{aligned} \tag{51}
$$

As in the one-loop case, the flow of these coupling constants does not depend on higher derivative couplings. The flow is qualitatively the same as the one-loop case depicted in Fig. 1. The fixed point corresponding to the dimensional reduction has the following correction

$$
R'(1) = \frac{\epsilon}{N-2} - \left(\frac{\epsilon}{N-2}\right)^2. \tag{52}
$$

The fixed-point equation for $R''(1)$ becomes a cubic algebraic equation. For the nonexistence of the solution $R''(1)$ in the perturbative region, this cubic equation has only one real solution. This condition requires the negative discriminant

$$
D = -\frac{4}{a^3}\left(c - \frac{b^2}{3a}\right)^3 - \frac{27}{a^2}\left(\frac{2b^3}{27a^2} - \frac{bc}{3a} + d\right)^2 < 0 \tag{53}
$$

for the cubic equation $aR''(1)^3 + bR''(1)^2 + cR''(1) + d = 0$. Expanding the left-hand side of the inequality (53) in $N - 18$ and $\epsilon = d - 4$, we obtain

$$
N < 18 - \frac{49}{5}\epsilon. \tag{54}
$$

This condition determines the region where the dimensional reduction breaks. The condition on N for the nonexistence of the solution $R''(1)$ is corrected from $N < 18$ in the two-loop order. This condition is identical to the existence condition of a suitable cuspy fixed point obtained by Le Doussal and Wiese,[12] which is consistent also with the phase diagram obtained by Tarjus and Tissier. In higher dimensions, dimensional reduction occurring is more likely.

C. Fixed point in the double expansion

The two-loop β function (50) enables us to calculate the higher-order corrections to the fixed point corresponding to dimensional reduction. For $N \geq 18 - \frac{49}{5}\epsilon$, this fixed point may exist, and it can be obtained in a double expansion with respect to $1/N$ and ϵ to several orders. We expand the fixed point up to necessary orders to discuss its stability.

$$
\begin{aligned}
R(z) =\ & \frac{\epsilon}{N}\left(z - \frac{1}{2}\right) + \frac{\epsilon}{N^2}\left(\frac{1}{2}z^2 + z\right) \\
& + \frac{\epsilon^2}{N^2}\left(-\frac{1}{2}z^2 - \frac{1}{2}\right) + \cdots.
\end{aligned} \tag{55}
$$

We study the stability of this fixed point in the scaling operator. In the two-loop analysis, we define the deviation function from this fixed point by

$$
\delta R'(z) = \frac{\epsilon}{N}v(z).
$$

We can calculate the correction of these higher orders to the eigenvalue equation for the deviation:

$$
\begin{aligned}
&(1-\epsilon)(1-z)^2(1+z)v''(z) \\
&+ [N - 4z - 2 + \epsilon(2+4z)](1-z)v'(z) \\
&+ [2(1-\epsilon)z - N\lambda]v(z) + [N - 2 + \epsilon(4-z)]v(1) = 0.
\end{aligned} \tag{56}
$$

As discussed in the large-N limit, we can determine the eigenvalues $\epsilon\lambda$. First, we study the equation for $v(1) = 0$. The solutions of this equation have regular singular points $z = 1$ and -1 for the interval $-1 \leq z \leq 1$.

Therefore, we can obtain the solutions in the following expansion forms:

$$v(z) = (1 - z)^\alpha \sum_{n=0}^{\infty} a_n (1 - z)^n \qquad (57)$$

around $z = 1$ and

$$v(z) = (1 + z)^\beta \sum_{n=0}^{\infty} b_n (1 + z)^n \qquad (58)$$

around $z = -1$. Substituting these forms into the eigenvalue equation (56), we require that the coefficients of the lowest order vanish. This requirement gives the indicial equations for the exponents α and β

$$2 (1 - \epsilon) \alpha^2 - (N - 4 + 4\epsilon) \alpha + 2 - 2\epsilon - N\lambda = 0, \quad (59)$$

$$2(1 - \epsilon)\beta^2 + N\beta = 0, \qquad (60)$$

which have the solutions

$$\alpha = \alpha_\pm(\lambda)$$
$$\equiv \frac{N - 4(1 - \epsilon) \pm \sqrt{N^2 - 8N(1 - \epsilon)(1 - \lambda)}}{4(1 - \epsilon)}, (61)$$

$$\beta = -\frac{N}{2(1 - \epsilon)} \quad \text{or} \quad 0. \qquad (62)$$

The coefficient of an arbitrary order satisfies the following recursion relation:

$$k[2(1 - \epsilon)(k + 2\alpha) - N + 4(1 - \epsilon)]a_k$$
$$-(1 - \epsilon)(\alpha + k)(\alpha + k + 1)a_{k-1} = 0,$$

for $k = 1, 2, 3, \ldots$. The solution of this recursion relation is

$$a_k = \frac{(1 + \alpha)_k (2 + \alpha)_k}{(3 + 2\alpha - N/(2 - 2\epsilon))_k} \frac{a_0}{2^k k!}, \qquad (63)$$

where the symbol $(\cdots)_k$ denotes

$$(x)_k = \Gamma(x + k)/\Gamma(x). \qquad (64)$$

The solutions can be expressed in terms of the Gaussian hypergeometric function

$$v(z, \alpha)$$
$$= (1 - z)^\alpha$$
$$\times {}_2F_1 \left(1 + \alpha, \ 2 + \alpha; \ 3 + 2\alpha - \frac{N}{2(1 - \epsilon)}; \ \frac{1 - z}{2} \right),$$

where the generalized hypergeometric function is defined by the following series expansion

$${}_mF_n (x_1, x_2, \ldots, x_m; y_1, y_2, \ldots, y_n; z)$$
$$\equiv \sum_{k=0}^{\infty} \frac{(x_1)_k (x_2)_k \cdots (x_m)_k}{(y_1)_k (y_2)_k \cdots (y_n)_k} \frac{z^k}{k!}. \qquad (65)$$

The general solution is given by the linear combination of the two independent solutions

$$v(z) = a_0^+ v(z, \alpha_+(\lambda)) + a_0^- v(z, \alpha_-(\lambda)), \qquad (66)$$

where $\alpha_\pm(\lambda)$ is defined by Eq. (61). In general, this solution has a singularity at $z = -1$ corresponding to $\beta = -\frac{N}{2(1-\epsilon)}$ given in Eq. (62). To obtain physical solutions, we eliminate the singularity at $z = -1$ by the suitable choice of $a_0^\pm$ for a requirement $|v(z)| < \infty$. Also the finiteness of the flow requires $\alpha_-(\lambda) = \frac{1}{2}$ or $\alpha_-(\lambda) \geq 1$ by the condition (23); then, we obtain the possible eigenvalues $\epsilon\lambda$ given by

$$\lambda = -\frac{1}{2} + \frac{9 - 9\epsilon}{2N} + \cdots, \qquad (67)$$

$$\lambda \leq -1 + \frac{8 - 8\epsilon}{N} + \cdots. \qquad (68)$$

Next we consider the case $v(1) \neq 0$. The existence of the solution given in the expansion requires vanishing the coefficient of $v(1)$, which determines λ:

$$\lambda = \lambda_1 = 1 + \frac{\epsilon}{N} + \cdots. \qquad (69)$$

The corresponding special solution is represented in terms of the generalized hypergeometric function

$$v_1(z)$$
$$= \frac{\epsilon}{2 - 2\epsilon} + \frac{2 - 3\epsilon}{2 - 2\epsilon}$$
$$\times {}_3F_2 \left(1, 1, 2, ; 1 - \alpha_+(\lambda_1), 1 - \alpha_-(\lambda_1); \frac{1 - z}{2} \right) (70)$$

The finite solution is given by a linear combination of this special solution $v_1(z)$ and a solution of the homogeneous equation with $v(1) = 0$ and with the λ given in Eq. (69):

$$v(z) = a_0^+ v(z, \alpha_+(\lambda_1)) + v_1(z). \qquad (71)$$

Since the coefficient a_0^+ should be chosen for the cancellation of the singularity at $z = -1$, this relevant mode is unique. Therefore, the fixed point is singly unstable; then, dimensional reduction can be observed in the critical exponents η and $\bar{\eta}$ for $N \geq 18 - \frac{49}{5}\epsilon$. This result agrees with a nonperturbative renormalization group obtained by Tarjus and Tissier[8] and a simple $1/N$ expansion.[10] All eigenfunctions are singular at $z = 1$. It is interesting that the eigenfunction (71) has an essential singularity at $1/N = 0$. This is because of the fact that we solve the eigenvalue equation nonperturbatively without a $1/N$ expansion after the derivation of the eigenvalue equation. Note that the two-loop renormalization group is useful to show the existence of the relevant mode. The limit $\epsilon \to 0$ of the relevant mode (70) corresponds to the eigenfunction of the one-loop scaling operator with the subleading correction in the $1/N$ expansion. The series expansion for the relevant mode seems to be ill defined, since

$$\lim_{\epsilon \to 0}[1 - \alpha_+(\lambda_1)] = 2 - \frac{N}{2}$$

is a negative integer for even $N \geq 6$. This apparent ill definition disappears by the two-loop or other higher-order corrections.

Here, we comment on the infinitely many relevant modes pointed out by Fisher.[5] To compare our result to Fisher's one-loop renormalization group analysis, we take the limit $\epsilon \to 0$ in the solution (61). They are included in the following series

$$\alpha_+(\lambda_k) = k - 1, \ (k = 3, 4, 5, \ldots) \quad \text{and} \quad a_0^- = 0.$$

These belong to the eigenvalues $\epsilon\lambda_k$ given by

$$\lambda_k = 1 - k + \frac{2k^2}{N} + \mathrm{O}\left(\frac{1}{N^2}\right),$$

which are positive for sufficiently large k. These agree with the eigenvalues (14) obtained by Fisher. Since these relevant modes diverge at $z = -1$, we have eliminated them as unphysical modes, as discussed above.

VI. HIGHER-ORDER CALCULATION

Here, we calculate higher-order correction in the $1/N$ expansion to the results. The fixed point corresponding to the dimensional reduction can be obtained in a double expansion with respect to $1/N$ and ϵ to several orders:

$$
\begin{aligned}
R(z) = & \ \frac{\epsilon}{N}\left(z - \frac{1}{2}\right) + \frac{\epsilon}{N^2}\left(\frac{1}{2}z^2 + z\right) \\
& + \frac{\epsilon^2}{N^2}\left(-\frac{1}{2}z^2 - \frac{1}{2}\right) \\
& + \frac{\epsilon}{N^3}\left(z^3 + \frac{5}{2}z^2 - 4z + \frac{9}{2}\right) \\
& + \frac{\epsilon^2}{N^3}\left(-\frac{7}{4}z^3 - \frac{15}{4}z^2 + \frac{35}{4}z - \frac{33}{4}\right) + \cdots
\end{aligned}
\tag{72}
$$

We study the stability of this fixed point in the scaling operator. We can calculate the correction of these higher orders to the eigenvalue equation for the deviation:

$$
\begin{aligned}
& -\frac{\epsilon}{N}(1-z)^3(1+z)^2 v'''(z) \\
& + \left[1 - \epsilon + \frac{1}{N}(3 + 2z) + \frac{3\epsilon}{2N}(1 + 3z)\right] \\
& \times (1-z)^2(1+z)v''(z) \\
& + \left[N - 4z - 2 + \epsilon(2 + 4z) - \frac{1}{N}(7 + 17z + 10z^2) \right. \\
& \left. + \frac{\epsilon}{4N}(31 + 27z + 18z^2)\right](1-z)v'(z) \\
& + \left[2(1-\epsilon)z + \frac{1}{N}(6z^2 + 8z) \right. \\
& \left. - \frac{\epsilon}{2N}(15z^2 + 21z + 2) - N\lambda\right] v(z) \\
& + \left[N - 2 + \epsilon(4 - z) - \frac{1}{N}(7 + 4z + 3z^2) \right. \\
& \left. + \frac{\epsilon}{N}\left(\frac{59}{4} + 7z - \frac{3z^2}{4}\right)\right] v(1) = 0.
\end{aligned}
\tag{73}
$$

As discussed in the previous section, we can determine the eigenvalues $\epsilon\lambda$. First we discuss the solution with $v(1) = 0$. To obtain finite solutions in the expansion (57) around $z = 1$ and (58) around $z = -1$, we should require that the coefficients of the lowest order vanish. These conditions give the indicial equations for the exponents α and β with two-loop corrections:

$$
\begin{aligned}
& \frac{4\epsilon}{N}\alpha^3 + 2\left(1 - \epsilon + \frac{5}{N}\right)\alpha^2 \\
& - \left(N - 4 + 4\epsilon - \frac{24}{N} + \frac{23\epsilon}{N}\right)\alpha \\
& + 2 - 2\epsilon + \frac{14}{N} - \frac{19\epsilon}{N} - N\lambda = 0, \\
& -\frac{8\epsilon}{N}\beta^3 + 4\left(1 - \epsilon + \frac{1}{N} + \frac{3\epsilon}{N}\right)\beta^2 \\
& + \left(2N - \frac{4}{N} + \frac{7\epsilon}{N}\right)\beta = 0.
\end{aligned}
\tag{74}
$$

The second equation has three solutions for $\beta = \beta_+ > 0$, $\beta = 0$, and $\beta = \beta_- < 0$. Since the solution with β_- diverges at $z = -1$, we should eliminate this solution. To this end, we need two independent solutions which are finite at $z = 1$. The existence of two solutions of $\alpha = \frac{1}{2}$ or $\alpha \geq 1$ requires the eigenvalue $\epsilon\lambda$ given by

$$\lambda = -\frac{1}{2} + \frac{9 - 9\epsilon}{2N} + \frac{57 - 60\epsilon}{2N^2} + \cdots, \tag{75}$$

$$\lambda \leq -1 + \frac{8 - 8\epsilon}{N} + \frac{48 - 38\epsilon}{N^2} + \cdots. \tag{76}$$

Next we consider the case $v(1) \neq 0$. The existence of the solution given in the expansion requires vanishing the coefficient of $v(1)$, which determines

$$\lambda = 1 + \frac{\epsilon}{N} + \frac{2\epsilon}{N^2} + \cdots. \tag{77}$$

The finite solution is given by a linear combination of this special solution of the inhomogeneous equation and a solution of the homogeneous equation with $v(1) = 0$ and with the λ given in Eq. (77). Therefore, the fixed point is singly unstable; then, dimensional reduction can be observed in the critical exponents η and $\bar{\eta}$ for $N \geq 18 - \frac{49}{5}\epsilon$.

VII. DISCUSSIONS

In this paper, we have studied the stability of the fixed points in the renormalization group for the random field O(N) spin model in $4 + \epsilon$ dimensions for sufficiently large N. We argue physical conditions on the random anisotropy function. This argument enables us to solve the eigenvalue problem for the infinitesimal deviation from all fixed points. In the large-N limit, all nontrivial fixed points are unstable or unphysical, except the fixed point corresponding to dimensional reduction.

We also discuss the stability of this fixed point on the basis of the two-loop renormalization group obtained by Le Doussal and Wiese[12] and Tissier and Tarjus.[13] These corrected results are essentially the same as those in the large-N limit. The fixed point corresponding to dimensional reduction cannot exist for $N < 18 - \frac{49}{5}\epsilon$. This condition on (d, N) agrees with the phase diagram obtained by Tarjus and Tissier[8] in a nonperturbative renormalization group and also with an existence condition of the cuspy fixed point obtained by the Le Doussal and Wiese.[12]

We derive the eigenvalue equation of the scaling operator in the double expansion in $1/N$ and ϵ on the basis of the two-loop renormalization group, and solve it nonperturbatively. We show that the unique singly unstable fixed point gives the critical exponents η and $\bar{\eta}$ predicted by dimensional reduction. This result agrees with that obtained by Tarjus and Tissier in the nonperturbative renormalization group[8] and also with the stability of the replica-symmetric saddle-point solution in the $1/N$ expansion.[10] Here, we summarize the physical insights obtained from our mathematical results for the random field O(N) spin model in $4 + \epsilon$ dimensions at zero temperature. There are two phases for sufficiently weak randomness. The fully stable fixed point $R(z) = 0$ makes the extended ferromagnetic phase. This phase is specified by the nonzero constant order parameter. The correlation of the spin fluctuation obeys the power law decay with the same exponent as that in the trivial Gaussian model. In the disordered phase, the order parameter vanishes and the spin correlation becomes short ranged. For $N \geq 18 - \frac{49}{5}\epsilon$, the phase transition between these two phases is governed by the singly unstable fixed point corresponding to dimensional reduction. Therefore, at any phase transition point on the phase boundary, the critical exponents predicted by dimensional reduction should be observed universally. On the other hand, for $N < 18 - \frac{49}{5}\epsilon$, the fixed point corresponding to dimensional reduction disappears. In this case, it is believed that a cuspy nonanalytic fixed point $R'(z) \sim (1 - z)^{1/2}$ governs the phase transition between the ferromagnetic and disordered phases. The breakdown of dimensional reduction is observed in the critical exponents.[6]

It is important to explore a breakdown of dimensional reduction in some other observables even for $N \geq 18 - \frac{49}{5}\epsilon$. For this problem, the statement on the fixed point by Tarjus and Tissier is interesting.[8] They argue that the singly unstable fixed point has a weak singularity

$$R'(z) \sim (1 - z)^{N/2}, \tag{78}$$

at $z = 1$ for large N. Even if the fixed point corresponding to dimensional reduction has this singularity, we cannot find such an essential singularity in the $1/N$ expansion. Therefore, it is possible that the obtained fixed point in the $1/N$ expansion is not analytic and has the weak singularity (78).

Acknowledgments

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[1] G. Parisi and N. Sourlas, Phys. Rev. Lett. **43**, 744 (1979).

[2] J. Z. Imbrie, Phys. Rev. Lett. **53**, 1747 (1984); Commun. Math. Phys. **98**, 145 (1985).

[3] J. Bricmont and A. Kupiainen, Phys. Rev. Lett. **59**, 1829 (1987); Commun. Math. Phys. **116**, 539 (1988).

[4] M. Aizenman and J. Wehr, Phys. Rev. Lett. **62**, 2503 (1989).

[5] D. S. Fisher, Phys. Rev. B **31**, 7233 (1985).

[6] D. E. Feldman, Phys. Rev. Lett. **88**, 177202 (2002).

[7] M. Mézard and A. P. Young, Europhys. Lett. **18**, 653 (1992).

[8] G. Tarjus and M. Tissier, Phys. Rev. Lett. **93**, 267008 (2004).

[9] J. R. L. de Almeida and D. Thouless, J. Phys. A **11**, 893 (1978).

[10] Y. Sakamoto, H. Mukaida, and C. Itoi, Phys. Rev. B **72**, 144405 (2005).

[11] L. Balents and D. S. Fisher, Phys. Rev. B **48**, 5949 (1993).

[12] P. Le Doussal and K. J. Wiese, Phys. Rev. Lett. **96**, 197202 (2006).

[13] M. Tissier and G. Tarjus, Phys. Rev. Lett. **96**, 087202 (2006).

[14] M. Schwartz and A. Soffer, Phys. Rev. Lett. **55**, 2499 (1985).